ON THE SOLUTION BY A VARIATIONAL METHOD OF THE PROBLEM OF THE OSCILLATIONS OF A LIQUID PARTIALLY FILLING A CAVITY

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PMM Vol.26, No.6, 1962, pp. 1122-1127

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(Received March 26, 1962)

The problem of the oscillation of an ideal liquid in a cavity is investigated by a variational method which enables us to obtain an approximate solution to any degree of accuracy for a cavity of arbitrary configuration.

It is shown that finding a solution to the boundary-value problem of the free oscillations of a liquid is equivalent to determining a system of functions for which the variation of the Hamilton functional assumes a minimum value.

The results obtained are used to determine the coefficients of the equations of disturbed motion for a solid body possessing a spherical cavity partially filled with an ideal liquid.

1. The reduction of the boundary-value problem of the oscillations of a liquid to a variational problem. We start from the assumption that the motion of the liquid can be defined by its kinetic energy T and its potential energy V.

Hamilton's variational principle states that the functional

$$J = \int_{t_1}^{t_2} (T - V)\, dt \tag{1.1}$$

must have a stationary value. In the present case

$$T = \frac{1}{2} \rho_0 \iiint_Q (\nabla \dot{U})^2\, dQ, \qquad V = \frac{1}{2} j \rho_0 \iint_\Sigma \varkappa^2\, ds \tag{1.2}$$

Therefore equality (1.1) means that in the interval between the times

t_1 and t_2 real motion must take place in such a way that the variation $\delta J = 0$, i.e. so that

$$\delta \int_{t_1}^{t_2} \left[\iiint_Q (\nabla \dot{U})^2 \, dQ - j \iint_\Sigma \varkappa^2 \, ds \right] dt = 0 \tag{1.3}$$

Here and in what follows Σ will denote the free surface, κ the equation of the free surface, ρ_0 the density of the liquid, U the potential of the displacements, Q the volume occupied by the liquid, j the acceleration of the mass forces, S the wetted surface.

Following the procedure outlined in [1], we introduce the potential of the displacements U of the boundary-value problem

$$\triangle U = 0 \quad (\text{within} Q), \qquad \frac{\partial U}{\partial \nu} = 0 \quad (\text{on } S), \qquad \frac{\partial^2 U}{\partial t^2} + j \frac{\partial U}{\partial \nu} = 0 \quad (\text{on } \Sigma) \tag{1.4}$$

where ν is the unit vector of the external normal. We express the potential U in the form of an infinite series

$$U(x, y, z, t) = \sum_{n=1}^{\infty} p_n(t) \zeta_n(x, y, z) \tag{1.5}$$

which within the volume Q and over a finite time interval $0 \leqslant t \leqslant t_k$ is absolutely and uniformly convergent.

We treat the functions $p_n(t)$ as coefficients of the expansion of U in a generalized Fourier series in functions ζ_n, which are orthogonal on the free surface and for which we take a normalization in the form

$$\frac{\partial \zeta_n}{\partial \nu} = 1 \qquad \begin{array}{c}\text{(at a point on the}\\ \text{contour } C)\end{array} \tag{1.6}$$

where C is the contour of the free surface.

The boundary-value problem (1.4) can be written in the form

$$\Delta \zeta_n = 0 \quad (\text{within} Q), \qquad \frac{\partial \zeta_n}{\partial \nu} = 0 \quad (\text{on } S), \qquad \frac{\partial \zeta_n}{\partial \nu} = \sigma_n^2 \zeta_n \quad (\text{on } \Sigma) \tag{1.7}$$

$$\sigma_n^2 = -\frac{1}{j} \frac{\ddot{p}_n(t)}{p_n(t)} = \text{const}$$

Applying Green's formula to expression (1.3) and making use of expansion (1.5), we make the stipulation that

$$\delta L = \delta \left[\iiint_Q \zeta_n \Delta \zeta_n \, dQ - \iint_S \zeta_n \frac{\partial \zeta_n}{\partial \nu} ds - \iint_\Sigma \zeta_n \left(\frac{\partial \zeta_n}{\partial \nu} - \sigma_n^2 \zeta_n \right) ds \right] = 0 \tag{1.8}$$

It follows from the basic principles of variational calculus that equality (1.8) is satisfied only for functions ζ_n which give the functional L a minimum. We will show that a function ζ_n, for which the functional $L[\zeta_n]$ assumes a minimum, is a solution to the problem (1.7).

Suppose that ζ_n gives the functional $L[\zeta_n]$ a minimum. We assess the value of the functional for the function $\zeta_n + \alpha\eta_n$, where η_n is a function which, together with its derivatives, is continuous within the volume Q. For small values of α we can take the function $\zeta_n + \alpha\eta_n$ as close to ζ_n as we like, and in addition

$$L[\zeta_n] \leqslant L[\zeta_n + \alpha\eta_n]$$

From the definition of the variation

$$\delta L = \left\{\frac{d}{d\alpha} L\,[\zeta_n + \alpha\eta_n]\right\}_{\alpha=0} = 0$$

we obtain

$$\iiint\limits_Q \eta_n \Delta\zeta_n \, dQ - \iint\limits_S \eta_n \frac{\partial \zeta_n}{\partial \nu} \, ds - \iint\limits_\Sigma \eta_n \left(\frac{\partial \zeta_n}{\partial \nu} - \sigma_n{}^2 \zeta_n\right) ds = 0 \qquad (1.9)$$

In accordance with the basic lemma of variational calculus [2], we find that equations (1.7) follow from (1.9), or in other words, the equations of small free oscillations of an ideal liquid are Euler equations of the functional L.

Thus, the process of solving the boundary-value problem (1.4) has been reduced to a variational problem which may be stated as follows: to find within a certain class of admissible functional arguments a function which is extremal for the functional L.

In order to solve this variational problem we make use of the method of Ritz, which is based on the construction of a minimizing sequence $\varphi^{(k)}$ such that

$$L[\varphi^{(k)}] = \text{minimum}$$

and which entails finding a complete system of functions $\gamma_1, \gamma_2, \ldots$, defined within the volume Q and possessing the property that linear combinations of the form

$$\varphi^{(k)} = c_1\gamma_1 + \ldots + c_k\gamma_k \qquad (1.10)$$

are admitted by the functional arguments, and that for $\varepsilon > 0$, no matter how small, we can find a value of k such that

$$|L[\zeta_n] - L[\varphi^{(k)}]| < \varepsilon$$

Consequently, assuming that $L[\varphi^{(k)}]$ is a continuous function of the parameters c_i, to find the minimum of the functional entails determining values of c_i such that

$$\frac{\partial L[\varphi^{(k)}]}{\partial c_i} = 0 \qquad (i = 1, \ldots, k) \tag{1.11}$$

We shall therefore try to find the kth approximation of the nth eigenfunction in the form (1.10), for which we obtain from (1.11) the system of equations

$$\frac{\partial L[\varphi^{(k)}]}{\partial c_i} = \iiint\limits_Q \nabla\gamma_i \nabla\varphi^{(k)}\, dQ - \sigma^2 \iint\limits_\Sigma \gamma_i \varphi^{(k)}\, ds = 0 \qquad (i = 1, \ldots, k)$$

which can conveniently be expressed in the form

$$\sum_{i=1}^{k} (a_{ij} - \sigma^2 b_{ij})\, c_i = 0 \qquad (j = 1, \ldots, k) \tag{1.12}$$

where

$$a_{ij} = \iiint\limits_Q \nabla\gamma_i \nabla\gamma_j\, dQ, \qquad b_{ij} = \iint\limits_\Sigma \gamma_i \gamma_j\, ds \tag{1.13}$$

The homogeneous system of k equations with k unknowns so obtained has a non-trivial solution when the determinant

$$D(\sigma^2) = |\alpha_{ij} - \sigma^2 b_{ij}| = 0 \qquad (i, j = 1, 2, \ldots, k) \tag{1.14}$$

The determinant (1.14) – an equation of the kth degree in σ^2 – gives k roots for σ_n^2 for each of which the system (1.12) has a non-zero solution $c_i^{(n)}$, which to the accuracy of a constant, determines the function $\varphi_n^{(k)}$ $(n = 1, \ldots, k)$. In order to determine $\varphi_n^{(k)}$ uniquely it is necessary to make use of the normalization (1.6) with $\varphi = \pi/2$.

On the other hand, the solution to the system of equations (1.12) is an approximate solution to the problem (1.7). Consequently, the values of the k roots of σ_n^2 found from (1.14) are none other than approximations to the first k eigenvalues $\sigma_1^2, \ldots, \sigma_k^2$, and the functions $\varphi_1^{(k)}, \ldots, \varphi_k^{(k)}$ are approximations for the corresponding eigenfunctions $\zeta_1, \ldots, \zeta_k$.

2. Solution for the case of a spherical cavity. The equations of motion for a solid body with a cavity partially filled with an ideal liquid have been derived in [1,3,4] and elsewhere.

Using the results of [1], we investigate the way in which the associated masses and the first frequency of natural oscillations depend on the depth of liquid, and on the basis of the results obtained we analyze the rate at which the process of successive approximations converges. In calculating the inertia coefficients of the liquid only the first mode of oscillations was taken into account.

The equations of disturbed motion for a solid body with a spherical cavity partially filled with liquid, when the forces exerted on the body by the liquid are calculated relative to the center of the sphere, are

$$(\mu + \mu^0)\ddot{\eta} + \sum_{n=1}^{\infty} \ddot{p}_n(t)\,\lambda_{2n} = P_{GZ}, \qquad \lambda_{2n} = \rho_0\sigma_n^2 \iint_{\Sigma} z\varphi_n ds$$

$$(J_{yy} + J_{yy}^0)\ddot{\vartheta} + \sum_{n=1}^{\infty} \ddot{p}_n(t)\,\lambda_{4n} = M_{GY}, \qquad \lambda_{4n} = \lambda_{2n}(X_G - X_c)$$

$$\mu_n \ddot{p}_n + j\sigma_n^2\mu_n p_n + \lambda_{2n}\ddot{\eta} + \lambda_{4n}\ddot{\vartheta} = 0, \quad \mu_n = \rho_0\sigma_n^2 \iint_{\Sigma} \varphi_n^2\, ds$$

$$(n = 1, 2, 3, \ldots) \tag{2.1}$$

Here η is the displacement of the body in a direction parallel to the Z-axis, ϑ is the angle of rotation of the body about an axis parallel to the y-axis; μ, μ^0 are the masses of the liquid and of the solid body, respectively, J_{yy}, J_{yy}^0 are the moments of inertia of the liquid and of the solid body respectively, x_c is the coordinate of the center of the sphere.

These equations describe the motion of the system comprising the solid body plus the liquid about the metacenter X_G, the position of which relative to the center of gravity of the system comprising the solid body plus the liquid in a solidified state is given by the formula

$$X_G = X_T + \frac{\pi\rho_0 R_\Sigma^4}{4(\mu + \mu^0)}$$

where R_Σ is the radius of the free surface.

It can easily be shown that for a spherical cavity the metacenter is at the center of the sphere. We introduce a spherical system of coordinates

$$x = R\cos\theta, \quad y = R\sin\theta\cos\varphi, \quad z = R\sin\theta\sin\varphi$$

with origin at the center of the sphere. The x-axis is perpendicular to the free surface Σ and lies in a direction opposite to the acceleration vector of the mass forces. We pass to the non-dimensional quantities

$$r = \frac{R}{R_0}, \quad l = \frac{h}{R_0}, \quad \omega_n^2 = R_0\sigma_n^2, \quad \alpha_{ij} = \frac{a_{ij}}{R_0^3}, \quad \beta_{ij} = \frac{b_{ij}}{R_0^4} \tag{2.2}$$

where R_0 is the radius of the sphere and h is the depth of liquid.

The elements of the characteristic determinant now become:

(a) for $l < 1$

$$\alpha_{ij} = \int_0^{2\pi} d\varphi \int_{\pi-\theta_0}^{\pi} \sin\theta d\theta \int_{r^*}^{1} \nabla\gamma_i \nabla\gamma_j r^2 dr \qquad \left(r^* = \frac{l-1}{\cos\theta}\right)$$

$$\beta_{ij} = (1-l)^2 \int_0^{2\pi} d\varphi \int_{\pi}^{\pi-\theta_0} [\gamma_i\gamma_j]_{r=r^*} \frac{\sin\theta}{\cos^3\theta}\, d\theta \qquad (\theta_0 = \arccos(1-l)) \tag{2.3}$$

(b) for $l > 1$

$$\alpha_{ij} = \int_0^{2\pi} d\varphi \int_0^{\theta_0} \sin\theta d\theta \int_0^{r^*} \nabla\gamma_i \nabla\gamma_j r^2 dr + \int_0^{2\pi} d\varphi \int_{\theta_0}^{\pi} \sin\theta d\theta \int_0^{1} \nabla\gamma_i \nabla\gamma_j r^2 dr$$

$$\beta_{ij} = (l-1)^2 \int_0^{2\pi} d\varphi \int_0^{\theta_0} [\gamma_i\gamma_j]_{r=r^*} \frac{\sin\theta}{\cos^3\theta}\, d\theta \qquad \left(r^* = \frac{l-1}{\cos\theta},\ \theta_0 = \arccos(l-1)\right)$$

As a minimizing sequence $\varphi^{(k)}$ we take a set of particular solutions of the Laplace equation which are harmonic within the sphere

$$\varphi^{(k)} = \sin\varphi \sum_{i=1}^{k} c_i r^i P_i^1(\cos\theta) \tag{2.4}$$

where $P_i^{\,1}$ $(\cos\theta)$ are associated Legendre functions of the first kind.

The sequence of functions $\varphi^{(k)}$ is chosen in the form of (2.4) from the following considerations:

(a) the first term of the expansion gives a frequency which coincides with that of the equivalent mathematical pendulum (the length of the pendulum is the distance from the center of the sphere to the center of mass of the undisturbed liquid). For small relative depths the frequency of such a pendulum is close to the frequency of the first mode of the liquid;

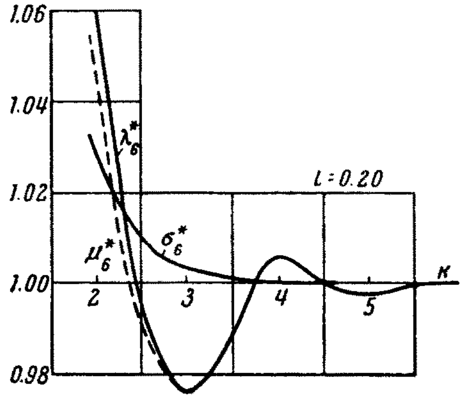

Fig. 1.

(b) the second term of the expansion is the potential of the displacements of the equivalent physical pendulum which, as will be shown later, for a certain range of depths can be taken as a good mechanical analogy for a liquid oscillating inside a sphere.

The calculations necessary to determine the natural frequency of the first mode of oscillations of the liquid and of the associated masses were carried out on an electronic digital computer. The program was arranged in such a way that the process of convergence to the exact value of any of the coefficients of the equations of motion continued until two consecutive results differed by less than 1%.

Figures 1 and 2, which are plotted from the results of these calculations, illustrate the rapidity of convergence of the coefficients of the equations of motion for two values of relative depths of liquid $l = h/R_0$. The numbers along the axis of abscissae denote the order of the characteristic determinant (the number of terms k taken into account in the expansion (2.4)); along the axis of ordinates are set out values of the ratios of any of the coefficients, calculated from a determinant of order k, to the corresponding coefficient taken as exact; for example

$$\mu_6{}^* = \frac{\mu_1^{(k)}}{\mu_1^{(6)}}, \qquad \lambda_6{}^* = \frac{\lambda_{21}^{(k)}}{\lambda_{21}^{(6)}}, \qquad \sigma_6{}^* = \frac{(\sigma_1^2)^{(k)}}{(\sigma_1^2)^{(6)}}$$

$$\mu_8{}^* = \frac{\mu_1^{(k)}}{\mu_1^{(8)}}, \qquad \lambda_8{}^* = \frac{\lambda_{21}^{(k)}}{\lambda_{21}^{(6)}}, \qquad \sigma_8{}^* = \frac{(\sigma_1^2)^{(k)}}{(\sigma_1^2)^{(8)}}$$

These figures show that for relative depths $l \leqslant 0.2$ the inertia characteristics of the liquid can be determined to an accuracy of approximately 5% by taking only the first two terms of the expansion, which is sufficient for practical purposes.

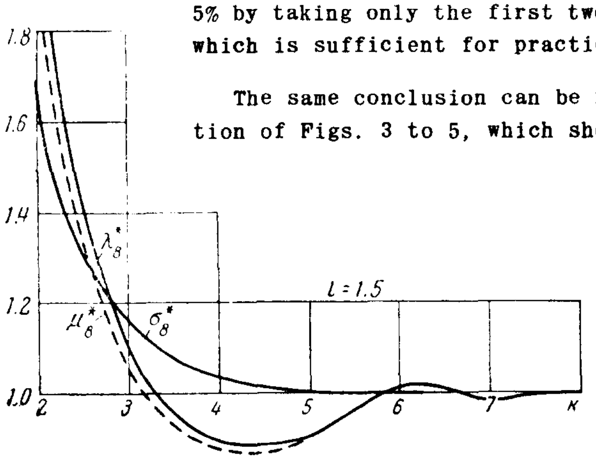

Fig. 2.

The same conclusion can be reached from a consideration of Figs. 3 to 5, which show a comparison of the results of calculations based on the variational method (curve *1*) with those obtained by other methods and by experiment [5]. In particular, a comparison is given of the coefficients of the equations of motion with the corresponding coefficients for an equivalent cylinder inscribed in the free surface (curve *2*), for equivalent physical (curve *3*) and mathematical (curve *4*) pendulums, and also with results obtained taking only the first two terms of the expansion (2.4) into account (curve *5*).

A comparison is also made with the results given in [6], in which the solution of the boundary-value problem for a sphere is obtained by means of approximate integral equations (curve *6*).

Figure 6 shows the general picture of the process of convergence of the first mode of natural oscillations.

The relation between λ_{21}, μ_1, σ_1^2, calculated to within an error of less than 1%, and the relative depth l is plotted in Fig. 7.

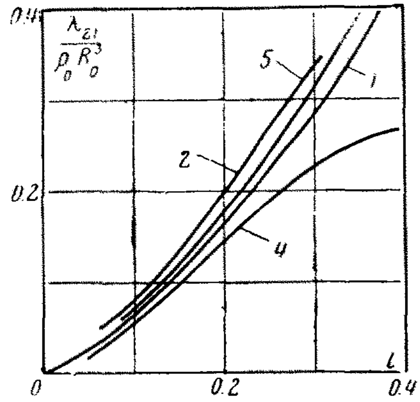

Fig. 3.

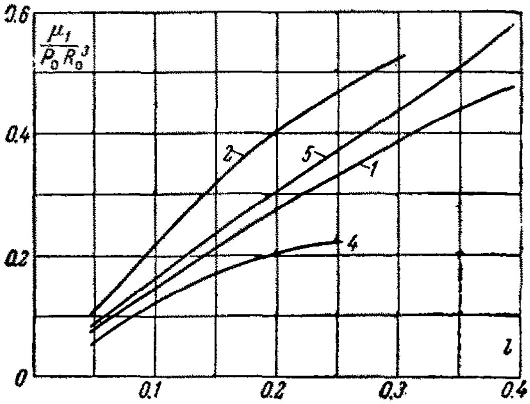

Fig. 4.

From the results obtained a number of conclusions can be drawn.

1. The equivalent pendulums are sufficiently close mechanical analogies for a liquid oscillating in a sphere with a relative depth $l < 0.1$.

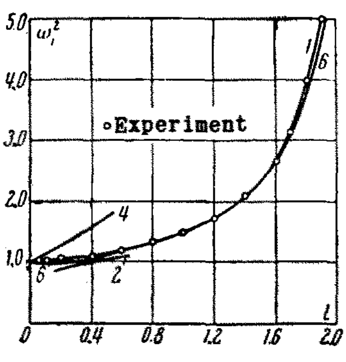

Fig. 5.

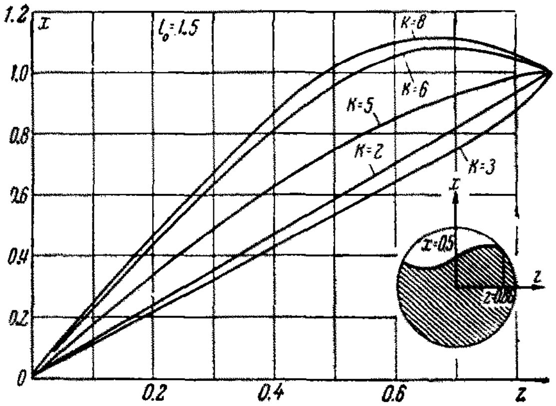

Fig. 6.

2. The errors in determining λ_{21}, μ_1, σ_1^2 associated with replacing the spherical cavity by an equivalent cylindrical cavity increase with decrease in the relative depth, and within the range of depths $0.5 < l <$ 0.4 give values of λ_{21} and μ_1 from 10 to 75% too large and values of σ_1^2

the same amount too small.

3. In order to determine the coefficients λ_{21}, μ_1 to a reasonable accuracy (to within an error of 1%) it is sufficient in the case of small relative depths to take the first 4 or 5 terms in the expansion (2.4); in the case of large relative depths 7 to 9 terms are sufficient. In order to determine the first natural frequency to the same accuracy the first 3 or 4 terms should be taken in the first case and 5 or 6 in the second.

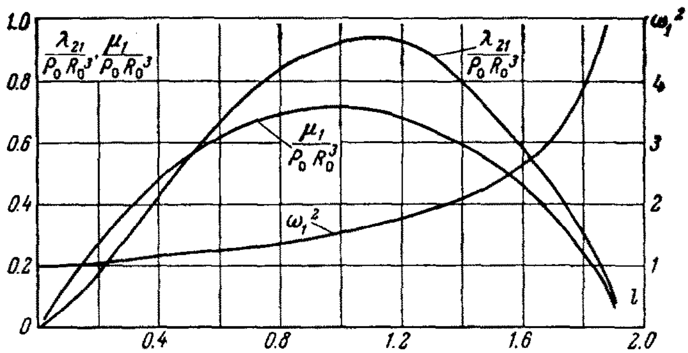

Fig. 7.

4. The results of calculations obtained by the variational method coincide fully with experimental results.

The author is indebted to Z.M. Poliakova, who was responsible for the programming and computing, and to B.I. Rabinovich under whose guidance the work was carried out.

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Translated by J.K.L.